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INTEGRALITY AND PRIME IDEALS IN FIXED RINGS OF P.I. RINGS

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Let R be a ring, G a finite group acting as automorphisms of R, and let R^G denote the fixed ring of G on R. When $|G|$, the order of G, is a unit in R, a number of facts are already known about the relationship between the primes of R and R^G, and about when R is integral over R^G [6,7,9,12]. In this note we show that some of these results can be improved when R satisfies a polynomial identity (PI).

We first consider integrality. If $|G|^{-1}\in R$ and G is abelian, D.S. Passman has shown that R is Schelter-integral over R^G [12]. This is false with no assumptions on $|G|$, even if R is a prime, affine, Noetherian PI ring [11]. We prove here that if R is a PI ring, and G is any finite group with $|G|^{-1}\in R$, then R is Schelter-integral over R^G.

We next consider prime ideals in R^G, still under the assumption that $|G|^{-1}\in R$. Following [9], two primes p,q of R^G are said to be equivalent if there exists a prime P of R such that p and q are both minimal over $P\cap R^G$. Various properties of this equivalence relation are known: equivalent primes have the same height [9] but not necessarily the same depth [7]. Also, the 'additivity principle' for Goldie ranks holds for primes in the extension $R\supset R^G$ [7]. We prove here that when R is a PI ring, and p and q are equivalent primes, then R^G/p and R^G/q have the same Gelfand–Kirillov dimension. In the special case that R is also Noetherian and affine, it follows that p and q have the same depth, a fact that was observed by J. Alev (private communication). As noted above, this is false in general. Moreover, we give an example of an affine Noetherian ring in which two equivalent primes have different depths.

Finally, we consider prime ideals in the case when $|G|$ is not assumed to be a unit. In particular, we consider a kind of incomparability question: if P, Q are primes of R with $P\cap R^G=Q\cap R^G$, must P and Q be in the same G-orbit of Spec(R)? This

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is known to be false in general [7]; we give here an example which shows it is false even for PI rings. However, if R is a semiprime PI ring, it is true if P and Q are identity faithful primes.

1. Integrality

We recall the notion of Schelter integrality [15]. For a ring S, $S \amalg_{\mathbb{Z}} \mathbb{Z}[x]$ denotes the coproduct (or free product) of S with the polynomial ring $\mathbb{Z}[x]$. Then the ring extension $R \supseteq S$ is *integral* if for every $r \in R$, there exists $p_r(x) \in S \amalg_{\mathbb{Z}} \mathbb{Z}[x]$ of the form $p_r(x) = x^n + q(x)$, where $q(x)$ has degree $<n$, so that $p_r(r)=0$.

Lemma 1. *Let $S \subseteq R$ be a ring extension. If I, J are ideals of R which are integral over S, then $I+J$ is integral over S.*

Proof. The proof is essentially the same as for classical integrality: use $(I+J)/J \cong I/(I \cap J)$, which is integral over S, and substitution.

Theorem 1. *Let R be a PI ring and G a finite group of automorphisms with $|G|^{-1} \in R$. Then R is integral over R^G.*

Proof. By Lemma 1, R has an ideal K maximal with respect to K being integral over R^G. For each $g \in G$, the image K^g of K under g is also integral over R^G; thus $\sum_{g \in G} K^g$ is integral over R^G and contains K. By the maximality of K, it follows that K is G-stable. Moreover K is a semiprime ideal, for otherwise there exists an ideal $N \supsetneq K$ with $N^2 \subseteq K$; but then N would be integral over R^G, a contradiction.

By passing to R/K, we may therefore assume that R is semiprime and has no non-zero ideals which are integral over R^G. We now use a theorem of Amitsur [1] as stated by Rowen [14, Theorem 1.4.21], which says that $\Delta_R \cdot R$ is integral (in the classical sense), over the center Z of R, where Δ_R denotes the subring of Z generated by evaluations of central polynomials. Now G acts on Z, which is integral over Z^G; thus $\Delta_R \cdot R$ is integral over $Z^G \subseteq R^G$. This contradicts our assumption that R has no integral ideals. Thus $R=K$, proving the theorem.

2. Equivalent primes, GK-dimension, and depth

As in Section 1, we assume that $|G|^{-1} \in R$. Recall from [9] that for each equivalence class $[p]$ of primes in $\mathrm{Spec}(R^G)$, there is a unique G-orbit of primes $\{P^g\}$ in R which determines the class; that is, $P \cap R^G = (\bigcap_g P^G) \cap R^G = p_1 \cap \cdots \cap p_n$, where $[p] = \{p_1, \ldots, p_n\}$.

For the definition and fundamental properties of Gelfand–Kirillov (GK) dimension, see [4].

Theorem 2. *Let R be a PI ring and G a finite group of automorphisms with $|G|^{-1} \in R$. If $p_1 \sim p_2$ in* $\mathrm{Spec}(R^G)$, *then* $\text{GK-dim}(R^G/p_1) = \text{GK-dim}(R^G/p_2)$.

Proof. Our argument is similar to the proof of the Theorem in [7]. Let $P \in \mathrm{Spec}(R)$ be as above so that $(\bigcap_g P^g) \cap R^G = p_1 \cap \cdots \cap p_n$. By passing to $R/\bigcap_g P^g$, we may assume that R is semiprime Goldie and that $\bigcap_i p_i = (0)$; then R^G is also Goldie.

Localizing, we have $Q(R) \cong \bigoplus_{g \in G} Q(R/P^g)$, a sum of isomorphic copies of $Q(R/P)$. Thus $Q(R)$ is 'GK-homogenous' in the sense of Borho [3], since all right ideals have the same GK-dim as $Q(R)$. Also, since $Q(R)$ is semi-simple Artinian and $|G|^{-1} \in R$, $Q(R)$ is a finite module over $Q(R)^G$ and $Q(R)^G = Q(R^G)$ [10]. By a result of Borho [3, §6.7], it follows that the minimal primes of $Q(R^G)$ have the same GK-dimension. But $Q(R^G) \cong \bigoplus_{i=1}^n Q(R^G/p_i)$, and so $\text{GK-dim}\, Q(R^G/p_1) = \text{GK-dim}\, Q(R^G/p_2)$. Finally, since R is a PI ring, all the quotient rings above are obtained by central localization. Since GK-dim is preserved under central localization, it follows that $\text{GK-dim}(R^G/p_1) = \text{GK-dim}(R^G/p_2)$.

We now obtain the result of Alev mentioned above,

Corollary. *Let R be a Noetherian PI algebra which is affine over a field k, and let G be a finite group of automorphisms of R with $|G|^{-1} \in R$. Then equivalent primes of R^G have the same depth.*

Proof. Since R is Noetherian and affine, R^G is also affine by a result of the present authors [11], so R^G/p_i is an affine PI-algebra. By a result of M.-P. Malliavin [8], the GK-dim of R^G/p_i is then equal to its classical Krull dimension. The corollary now follows from Theorem 2.

The next example shows that the PI hypothesis is necessary.

Example 1. An affine Noetherian ring R, with a group G of order 2, with $1/2 \in R$, but two equivalent primes in R^G of different depths.

Proof. Let $A = k[x, y \mid xy - yx = 1]$ be the first Weyl algebra over a field k of characteristic 0. Let $\lambda = Ax$, a left ideal, and let $\Pi = k + Ax$. By [13, Theorem 7.4], Π is (left) Noetherian.

Now let

$$R = \begin{pmatrix} \Pi & A \\ Ax & A \end{pmatrix},$$

that is, the subring of $M_2(A)$ with entries from the appropriate subrings of A. We claim that R is an affine, Noetherian k-algebra. R is Noetherian since it is a finite (left and right) module over the diagonal $\Pi \oplus A$, and both Π and A are Noetherian. To see that R is affine, one can verify that the following elements generate R over k:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0\\ x & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}, \text{ and } \begin{pmatrix}0 & 0\\ 0 & y\end{pmatrix}.$$

We also remark that Π is affine: this follows by [11, Corollary 1] since $\Pi \cong e_{11}Re_{11}$.

Now let g be conjugation by $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; then $g^2=1$, and letting $G=\langle g\rangle$,

$$R^G = \begin{pmatrix}\Pi & 0\\ 0 & A\end{pmatrix} \cong \Pi \oplus A.$$

Let $p_1=(0,A)$ and let $p_2=(\Pi,0)$. Then p_1 and p_2 are primes of R^G, and $p_1 \sim p_2$ since they are both minimal over $(0)=(0)\cap R^G$. Since A is simple, p_2 is maximal. However p_1 is not maximal since λ is a proper ideal of Π.

3. G-orbits of prime ideals

For a commutative ring R, and any finite group G of automorphisms, it is well-known that if P,Q are primes in R with $P\cap R^G=Q\cap R^G$, then $Q=P^g$ for some $g\in G$; an elementary proof appears in Bourbaki [5, Ch. 5, §2, Theorem 2]. For non-commutative rings, this is false in general [7], although true if $|G|^{-1}\in R$. We sketch a proof of this fact for completeness. Let $A=\bigcap_{g\in G} P^g$ and $B=\bigcap_{g\in G} Q^g$; then it suffices to show that $A=B$ (for then each P^g contains some Q^h and each Q^h contains some P^k; as G is finite, repeating the procedure gives $Q=P^g$, some g). Say that $A\subsetneq B$; then in the semiprime ring $\bar{R}=R/B$, $\bar{A}\neq 0$, but $\bar{A}^G=\overline{A\cap R^G}=\overline{B\cap R^G}=\overline{(0)}$. This contradicts the theorem of Bergman and Isaacs [2]. Thus, when $|G|^{-1}\in R$, G is transitive on the primes having a common intersection with R^G.

We note that although not explicitly stated, the above fact is implicit in [6].

When R is semiprime PI, something can be said even if $|G|^{-1}\notin G$. The next lemma acts as a substitute for the Bergman–Isaacs Theorem [2].

Lemma 2. *Let R be semiprime PI and G finite. If I is a non-zero G-stable ideal of R, then $I\cap R^G\neq(0)$.*

Proof. I itself is a semiprime PI ring, and so has non-trivial center C [14]. Now G acts on C, a commutative ring with no nilpotent elements; thus $C^G\neq 0$, as C is integral over C^G. But $C^G\subseteq I\cap R^G$.

Recall that a prime P of R is *identity-faithful* if $\operatorname{pi}\deg(R/P)=\operatorname{pi}\deg(R)$.

Theorem 3. *Let R be a semiprime PI ring, G finite, and P,Q primes of R with $P\cap R^G=Q\cap R^G$. Then P and Q are in the same G-orbit in either of the following situations:*

(1) *Either P or Q is identity-faithful.*

(2) $P\cap R^G=Q\cap R^G=(0)$.
In the second situation, $\bigcap_g P^g=(0)$, *and so R has* $\le|G|$ *minimal primes.*

Proof. As in the argument for the case of $|G|^{-1}\in R$ above, we let $A=\bigcap_{g\in G}P^g$ and $B=\bigcap_{g\in G}Q^g$; it suffices to show $A=B$.

In case (2), this follows from Lemma 2: A and B are G-stable ideals with $A\cap R^G=B\cap R^G=(0)$, and so $A=B=(0)$.

For case (1), assume that Q is identity-faithful. We use Amitsur's result as in the proof of Theorem 1 to get that $\Delta_A\cdot A$ is integral over A^G. Since $A^G=B^G$ and B is an ideal, it follows that for any $a\in\Delta_A\cdot A$ there exists $n>0$ with $a^n\in B$. But R/B is a semiprime PI ring, so has no non-zero nil ideals. Thus $\Delta_A\cdot A\subseteq B\subseteq Q$. Since Q is prime, either $A\subseteq Q$ or $\Delta_A\subseteq Q$. Now if $A\not\subseteq Q$, then $\Delta_A\subseteq Q$ and so in R/Q, a prime ring with the same p.i. degree as R, the image of A satisfies identities of lower degree, a contradiction. Thus $A\subseteq Q$. Since A is G-stable, $A\subseteq\bigcap_g Q^g=B$.

Moreover, $A=\bigcap P^g\subseteq Q$ implies that for some $g, P^g\subseteq Q$. But then P^g, and so P, is also identity faithful. Repeating the above argument, we see $B\subseteq A$. Thus $B=A$ and the theorem is proved.

We do not know an example of a semiprime PI ring R, such that for two primes P,Q with $P\cap R^G=Q\cap R^G$, P and Q are not in the same G-orbit. However, as the following example shows, this can fail when R is not semiprime, even when R is finite over its center.

Example 2. A PI ring R, and finite group G, such that R has two G-stable primes P,Q with $P\cap R^G=Q\cap R^G$.

Proof. We let k be a field of characteristic 2, and let $A=k\oplus k$. Identify k with the diagonal $\{(\alpha,\alpha)\}\subseteq A$, and consider

$$R=\begin{pmatrix}k & A\\ 0 & A\end{pmatrix}.$$

R is certainly a finite module over its center. Let g be conjugation by $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$. Then if $G=\langle g\rangle$,

$$R^G=\left\{\begin{pmatrix}\alpha & a\\ 0 & (\alpha,\alpha)\end{pmatrix}\,\middle|\, \alpha\in k, a\in A\right\}.$$

Let

$$P=\begin{pmatrix}k & A\\ 0 & (k,0)\end{pmatrix}\quad\text{and}\quad Q=\begin{pmatrix}k & A\\ 0 & (0,k)\end{pmatrix}.$$

Then P and Q are primes with $P\cap R^G=Q\cap R^G=\left(\begin{smallmatrix}0&A\\0&0\end{smallmatrix}\right)$. However, $P^g=P$ and $Q^g=Q$.

Note in this example that when we pass to $\bar{R}=R/N$, where N is the nil radical, the induced action of G on $\bar{R}$ is trivial. Thus $\bar{R}^G=\bar{R}$ and $\bar{P}\cap\bar{R}^G\neq\bar{Q}\cap\bar{R}^G$. Thus the hypothesis on the primes is lost in homomorphic images.

Note added in proof

The Corollary to Theorem 2 has been improved in several ways. First, J. Alev has shown that it holds if R is an affine PI ring, not necessarily Noetherian, in "Sur l'extension $R^G \hookrightarrow R$", Comm. Algebra, to appear.

Second, the present authors have realized that the conclusion of the Corollary holds if R is a Noetherian PI ring, not necessarily affine, by an argument similar to the proof of Theorem 2. However, depth is not an equivalence relation for any PI ring with $|G|^{-1} \in R$; this is seen by an example based on Example 1. The details of these two facts will appear in: S. Montgomery, "Group actions on rings: some classical problems", Proceedings of Nato A.S.I. in Ring Theory, Antwerp, 1983.

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